L^∞-Stability of IMEX-BDF2 Finite Volume Scheme for Convection-Diffusion Equation

Caterina Calgaro, Meriem Ezzoug

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Abstract In this paper, we propose a finite volume scheme for solving a two-dimensional convection-diffusion equation on general meshes. This work is based on a implicit-explicit (IMEX) second order method and it is issued from the seminal paper [2]. In the framework of MUSCL methods, we will prove that the local maximum property is guaranteed under an explicit Courant-Friedrichs-Levy condition and the classical hypothesis for the triangulation of the domain.

Key words: Convection-diffusion equation, finite volume scheme, IMEX-BDF2 scheme, L^∞-stability

MSC (2010): 65M99, 76M12, 76E17

1 Introduction

Convection-diffusion processes appear in many areas of science, e.g. fluid dynamics or heat and mass transfer. In the study of the evolution of a mixture, the system of PDEs derives from the compressible Navier-Stokes equations. The mixture of two viscous fluids is described by the density $\rho \geq 0$, the *mass velocity* field $\mathbf{v}$ (which is not solenoidal) and the pressure p. Following Kazhikhov and Smagulov [11], we set

$$\mathbf{u} = \mathbf{v} + \lambda \nabla \ln(\rho), \tag{1}$$

Caterina Calgaro
Univ. Lille, CNRS, UMR 8524 - Laboratoire Paul Painlevé,
F-59000 Lille, France
e-mail: caterina.calgaro@univ-lille1.fr

Meriem Ezzoug
UR: Multifractals et Ondelettes. FSM. Université de Monastir. 5019 Monastir, Tunisie
ISSIG. Université de Gabès. 6032 Gabès, Tunisie
e-mail: meriemezzoug@yahoo.fr

for some mass diffusion coefficient $\lambda > 0$. This Fick's law describes the diffusive fluxes of one fluid into the other. Clearly, the *volume velocity* field $\mathbf{u}$ satisfies $\mathrm{div}\,\mathbf{u} = 0$ and we obtain the non-standard constraint $\mathrm{div}\,\mathbf{v} = -\mathrm{div}\big(\lambda \nabla \ln(\rho)\big)$, which is relied on the definition of the pressure p. Using (1), the mass conservation equation becomes

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{u}) = \lambda\, \Delta \rho. \tag{2}$$

The momentum equation can also be rewritten in order to obtain the Kazhikhov-Smagulov model [11]. This model was firstly studied in [14, 1] (see also references therein). The mathematical analysis in a three-dimensional domain of Kazhikhov-Smagulov type models was carried out in recent works [5, 8], where the authors study the Kazhikhov-Smagulov models with a specific Korteweg stress tensor. The numerical study of a Kazhikhov-Smagulov model for the two-dimensional case can be found in [4], where the authors propose an hybrid finite volume-finite element method combined with the backward Euler method in time. In order to generalize the analysis given in [4] to second-order methods in time and space, the first goal is to recover the L^∞-stability of the finite volume method used for the convection-diffusion equation. This is the purpose of this paper.

2 Description of the numerical scheme

This section is devoted to the design of a numerical scheme to approximate (2), using the vertex-based MUSCL finite volume methods introduced in [13] and used in [2] for a second-order accuracy in space, and an implicit-explicit (IMEX) linear multistep methods [10] for a second-order in time.

Mesh definitions and notations. Let Ω be an open bounded polygonal subset on $\mathbb{R}^2$, with sufficiently regular boundary $\partial\Omega$, and $[0,T]$ the time interval, for $T > 0$. The discretization of (2) will be carried out on an unstructured triangular mesh. We denote by $\mathscr{T}_h$ a partition of Ω composed of conforming and isotropic triangles T_k, $k \in [1,K]$, with $K \in \mathbb{N}^*$. The $\mathscr{T}_h$ is called the *primal mesh*. We suppose the following hypotheses:

(H1) Let $\{\mathscr{T}_h\}_{h>0}$ be a *regular* family of triangulations of Ω.
(H2) The triangulation $\mathscr{T}_h$ is of *weakly acute type* (no triangle with an angle greater than $\pi/2$).

For each element $T \in \mathscr{T}_h$, we denote B_T the barycenter of the triangle, $|T|$ the area of T, and M_i, M_{j_1}, M_{j_2} the three vertices of T. We also denote respectively M_{ij_1} and M_{ij_2} the middles of $[M_iM_{j_1}]$ and $[M_iM_{j_2}]$.
Let us construct the *dual mesh* $\mathscr{C}_h = \{\mathscr{C}_i, i \in [1,I]\}$, which defines a second partition of Ω, ($I \in \mathbb{N}^*$ is the number of vertices of $\mathscr{T}_h$). The dual finite volume $\mathscr{C}_i$ associated with each vertex M_i, $i \in [1,I]$, is a closed polygon obtained in the following way: we join the barycenter B_T of every triangle $T \in \mathscr{T}_h$ which share the vertex M_i with

the middle point of every side of T containing M_i (see Fig. 1). If $M_i \in \partial\Omega$, then we complete the boundary of $\mathscr{C}_i$ by the segments joining M_i with the middle point of boundary sides that contain M_i. $\mathscr{C}_i$ is often called the vertex-based control volume around the node M_i. Accordingly, we have $\bigcup_{T\in\mathscr{T}_h} T = \overline{\Omega} = \bigcup_{i\in[1,I]} \mathscr{C}_i$.

Moreover, if we denote $|\mathscr{C}_i|$ the area of $\mathscr{C}_i \in \mathscr{C}_h$, then $|\mathscr{C}_i| = \sum_{T,M_i\in T} \dfrac{|T|}{3}$.

For $i \in [1,I]$, let $\mathscr{V}(i) = \left\{ j \in [1,I], \mathscr{C}_j \text{ is a neighbor of } \mathscr{C}_i \right\}$. For $l = 1,2$, we denote $\Gamma_{ij_l}^{(T)}$ the segment $[M_{ij_l}B_T]$, $A_{ij_l}^{(T)}$ its middle point, $\mathbf{n}_{ij_l}^{(T)}$ the unit outward normal to $\mathscr{C}_i$ along $\Gamma_{ij_l}^{(T)}$ and $|\Gamma_{ij_l}^{(T)}|$ the length of $\Gamma_{ij_l}^{(T)}$. For $T \in \mathscr{T}_h$ and $M_i \in T$, we have:

$$\sum_{l=1}^{2} |\Gamma_{ij_l}^{(T)}| \mathbf{n}_{ij_l}^{(T)} = -|T| \nabla \psi_i, \tag{3}$$

where ψ_i is the $\mathbb{P}_1$ basis function associated to the vertex M_i of T. For every $\mathscr{C}_i \in \mathscr{C}_h$, the boundary of $\mathscr{C}_i$ is

$$\partial\mathscr{C}_i = \bigcup_{T,M_i\in T} \left(\Gamma_{ij_1}^{(T)} \cup \Gamma_{ij_2}^{(T)} \right). \tag{4}$$

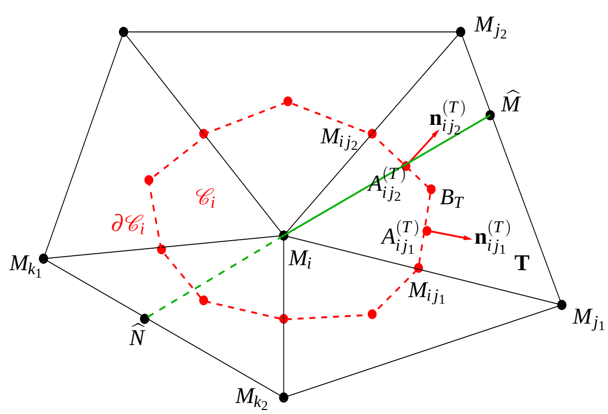

Fig. 1 Dual mesh-Vertex based control volume $\mathscr{C}_i$ around the node M_i.

IMEX-BDF2 finite volume scheme. Here, we describe the finite volume scheme for solving (2). In order to obtain the density reconstruction on the interfaces $\Gamma_{ij_l}^{(T)}$, we use the MUSCL technique with a multislope gradient reconstruction. Concerning the time discretization, we adapt the implicit-explicit (IMEX) linear 2-step methods using *extrapolated BDF2* scheme for the convective term combined with *implicit BDF2* scheme for the diffusive term. The velocity field $\mathbf{u}(t,\mathbf{x}) \in \mathbb{R}^2$ is a given function verifying the divergence free condition. For the space discretization, the usual vertex-based finite volume scheme on control volume $\mathscr{C}_i$, for all $i \in [1,I]$, reads

$$\frac{d}{dt}\int_{\mathscr{C}_i}\rho(t,\mathbf{x})\,d\mathbf{x}+\int_{\partial\mathscr{C}_i}\rho(t,\mathbf{x})\mathbf{u}(t,\mathbf{x})\cdot\mathbf{n}\,d\sigma=\lambda\int_{\partial\mathscr{C}_i}\nabla\rho(t,\mathbf{x})\cdot\mathbf{n}\,d\sigma. \tag{5}$$

We denote by Δt the time step and $t^n = n\Delta t$, $n \geq 0$, but variable time steps can also be used. Then, the approximate solution ρ_i^n, $i \in [1,I]$, at time t^n, verifies

$$\rho_i^n \approx \frac{1}{|\mathscr{C}_i|}\int_{\mathscr{C}_i}\rho(t^n,\mathbf{x})\,d\mathbf{x}.$$

In particular, the numerical approximation of the density is a piecewise constant function in space on the control volume $\mathscr{C}_i$. For the time discretization, we consider the implicit BDF2 scheme and an extrapolated BDF2 scheme following [10]. With this choice, we obtain a second-order accuracy in time. Then, the equation (5) is rewritten as follows, for each $i \in [1,I]$ and $n \geq 1$,

$$\begin{aligned}\rho_i^{n+1}-\frac{2\lambda}{3}\frac{\Delta t}{|\mathscr{C}_i|}\int_{\partial\mathscr{C}_i}\nabla\rho(t^{n+1},\mathbf{x})\cdot\mathbf{n}\,d\sigma &= \frac{4}{3}\rho_i^n-\frac{4}{3}\frac{\Delta t}{|\mathscr{C}_i|}\int_{\partial\mathscr{C}_i}\rho(t^n,\mathbf{x})\mathbf{u}(t^n,\mathbf{x})\cdot\mathbf{n}\,d\sigma\\ &-\frac{1}{3}\rho_i^{n-1}+\frac{2}{3}\frac{\Delta t}{|\mathscr{C}_i|}\int_{\partial\mathscr{C}_i}\rho(t^{n-1},\mathbf{x})\mathbf{u}(t^{n-1},\mathbf{x})\cdot\mathbf{n}\,d\sigma.\end{aligned} \tag{6}$$

In order to approximate $\nabla\rho(t^{n+1},\mathbf{x})$ in (6), we consider a $\mathbb{P}_1$-finite element approach for the density such that

$$\rho^{n+1}_{|_T} \approx \sum_{M_j\in T}\psi_j\,\rho_j^{n+1}, \qquad \text{for all } T\in\mathscr{T}_h,$$

with $\{\psi_j\}_{j\in[1,I]}$ the canonical basis of the usual $\mathbb{P}_1$ finite element space. Using (3) and (4), we find ρ_i^{n+1}, $i \in [1,I]$, $n \geq 1$, verifying the following second-order *IMEX-BDF2 finite volume scheme*:

$$\begin{aligned}\rho_i^{n+1}+\frac{2\lambda}{3}\,\frac{\Delta t}{|\mathscr{C}_i|}\sum_{T,M_i\in T}|T|\sum_{M_j\in T}\nabla\psi_i\cdot\nabla\psi_j\;\rho_j^{n+1}&\\ =\;\frac{4}{3}\,\rho_i^n-\frac{4}{3}\,\frac{\Delta t}{|\mathscr{C}_i|}\sum_{T,M_i\in T}\sum_{l=1}^{2}|\Gamma_{ij_l}^{(T)}|\;G_{ij_l}^n\big(\rho_{ij_l}^n,\rho_{j_li}^n\big)&\\ -\;\frac{1}{3}\,\rho_i^{n-1}+\frac{2}{3}\,\frac{\Delta t}{|\mathscr{C}_i|}\sum_{T,M_i\in T}\sum_{l=1}^{2}|\Gamma_{ij_l}^{(T)}|\;G_{ij_l}^{n-1}\big(\rho_{ij_l}^{n-1},\rho_{j_li}^{n-1}\big).&\end{aligned} \tag{7}$$

Here we denote by $G_{ij_l}\big(\rho_1,\rho_2\big)$ a numerical flux that satisfies the consistency, conservativity and monotonicity properties. In particular, for any constant function ρ_1, we have

$$\sum_{k\in\mathscr{V}(i)}|\Gamma_{ik}^{(T)}|\;G_{ik}\big(\rho_1,\rho_1\big)=0. \tag{8}$$

In [2], G_{ij_l} is the upstream flux, but many other numerical fluxes can be considered, as for instance Lax-Friedrichs or Engquist-Osher fluxes. We underline that for multi-

physics coupled models, a particular attention must be paid in the approximation of the continuous velocity associated to any point of $\partial\mathscr{C}_i$ (see [3]).
In (7), ρ_{ij_l} and ρ_{j_li} denote the density reconstructions on the segments $\Gamma_{ij_l}^{(T)}$, for $l=1,2$. In order to reach a second-order accuracy in space, we use the MUSCL technique [13] with a multislope gradient reconstruction. Introducing $\widehat{M}=[M_{j_1}M_{j_2}]\cap(M_iA_{ij_l}^{(T)})$ and $\widehat{N}\in[M_{k_1}M_{k_2}]\cap(M_iA_{ij_l}^{(T)})$, we define

$$p_{ij_l}^{\text{up}}=\frac{\rho_i-\rho_{\widehat{N}}}{\|M_i\widehat{N}\|}\quad\text{and}\quad p_{ij_l}^{\text{down}}=\frac{\rho_{\widehat{M}}-\rho_i}{\|M_i\widehat{M}\|}.$$

Then, ρ_{ij_l} is the density evaluated at node $A_{ij_l}^{(T)}$, defined as:

$$\rho_{ij_l}=\rho_i+\widehat{p}_{ij_l}\,\|M_iA_{ij_l}^{(T)}\|,\quad\text{with}\quad\widehat{p}_{ij_l}=p_{ij_l}^{\text{up}}\mathrm{Lim}\Big(\frac{p_{ij_l}^{\text{down}}}{p_{ij_l}^{\text{up}}}\Big),$$

where Lim is a so-called "τ-limiter" (for details see [2]). In particular, they have the following result:

Lemma 1. *There exists some coefficients $\omega_{ij_lk}\geq 0$, $k\in\mathscr{V}(i)$, such that*

$$\rho_{ij_l}-\rho_i=\sum_{k\in\mathscr{V}(i)}\omega_{ij_lk}\big(\rho_i-\rho_k\big)$$

holds, and furthermore, they verify $\sum_{k\in\mathscr{V}(i)}\omega_{ij_lk}\leq\frac{7\tau}{12}C_{\mathscr{T}_h}$, where the constant $C_{\mathscr{T}_h}$ characterizes the mesh regularity (but it is more general than the classical Ciarlet ratio) and $\tau>0$ is used in the definition of the τ-limiter.

3 L^∞-stability of the numerical scheme

The IMEX-BDF2 finite volume scheme (7) is rewritten as linear system:

$$\mathbf{A}\,\rho^{n+1}=\mathbf{F}^n,\tag{9}$$

where the matrix $\mathbf{A}$ and the right hand side $\mathbf{F}^n$ are defined as follows:

$$\begin{aligned}
A_{i,i}\;&=1+2\lambda\Delta t\sum_{T,M_i\in T}\|\nabla\psi_i\|^2,\quad A_{i,j}=2\lambda\Delta t\sum_{T,M_i\neq M_j\in T}\nabla\psi_i\cdot\nabla\psi_j,\quad\forall i,j\in[1,I],\\
F_i^n\;&=\frac{4}{3}\,\rho_i^n-\frac{4}{3}\frac{\Delta t}{|\mathscr{C}_i|}\sum_{T,M_i\in T}\sum_{l=1}^{2}|\Gamma_{ij_l}^{(T)}|\;G_{ij_l}^n\big(\rho_{ij_l}^n,\rho_{j_li}^n\big)\\
&\quad-\frac{1}{3}\,\rho_i^{n-1}+\frac{2}{3}\frac{\Delta t}{|\mathscr{C}_i|}\sum_{T,M_i\in T}\sum_{l=1}^{2}|\Gamma_{ij_l}^{(T)}|\;G_{ij_l}^{n-1}\big(\rho_{ij_l}^{n-1},\rho_{j_li}^{n-1}\big),\qquad\forall i\in[1,I].
\end{aligned}$$

Under the hypotheses (H1) and (H2) on the mesh $\mathscr{T}_h$, the matrix $\mathbf{A}$ is an M-matrix.

Remark 1. The hypothesis (H2), necessary to establish error estimates, is classical for the vertex-centered finite volume scheme [12] or the combined finite volume-finite element scheme [9]. Obviously, the M-matrix property still holds for Delaunay triangulations (see [6], Sect. 3.4).

Now, we prove the following result:

Proposition 1. *If for any $n \geq 1$, we have $\rho^{n-1} \geq 0$ and $\rho^n \geq 0$, then the right hand side of linear system (9) satisfy $\mathbf{F}^n \ \geq \ 0$ under the CFL condition:*

$$\Delta t \ \leq \ \min_{1\leq i\leq I} \frac{|\mathscr{C}_i|}{\dfrac{2}{3}\Big(\dfrac{7\tau}{12} C_{\mathscr{T}_h} + 2\Big) \parallel \mathbf{u} \parallel_{i,\infty} \displaystyle\sum_{T,M_i\in T} \Big(|\Gamma_{ij_1}^{(T)}| + |\Gamma_{ij_2}^{(T)}|\Big)}, \tag{10}$$

with $\parallel \mathbf{u} \parallel_{i,\infty} = \max\limits_{T,M_i\in T} \parallel \mathbf{u}_T \parallel_{l^2(\mathbb{R}^2)}$ *where* $\mathbf{u}_T$ *is the cell average velocity.*

Proof. Let $i \in [1,I]$ and $n \geq 1$. Thanks to (8), the i-th row of (9) is given by:

$$\begin{aligned}\Big(\mathbf{A}\rho^{n+1}\Big)_i &= \frac{4}{3}\,\rho_i^n - \frac{4}{3}\frac{\Delta t}{|\mathscr{C}_i|}\sum_{T,M_i\in T}\sum_{l=1}^{2}|\Gamma_{ij_l}^{(T)}|\Big(G_{ij_l}^n\big(\rho_{ij_l}^n,\rho_{j_li}^n\big) - G_{ij_l}^n\big(\rho_i^n,\rho_i^n\big)\Big)\\ &- \frac{1}{3}\,\rho_i^{n-1} + \frac{2}{3}\frac{\Delta t}{|\mathscr{C}_i|}\sum_{T,M_i\in T}\sum_{l=1}^{2}|\Gamma_{ij_l}^{(T)}|\Big(G_{ij_l}^{n-1}\big(\rho_{ij_l}^{n-1},\rho_{j_li}^{n-1}\big) - G_{ij_l}^{n-1}\big(\rho_i^{n-1},\rho_i^{n-1}\big)\Big).\end{aligned} \tag{11}$$

Let us introduce some definitions and notations dropping the time indices, such that

$$\Delta\rho_{ij_l} = \rho_{ij_l} - \rho_i, \quad \widetilde{\Delta}\rho_{ij_l} = \rho_{j_li} - \rho_i, \quad \text{for } l = 1,2.$$

Thanks to Lemma 1, there exists for $l = 1,2$, some coefficients $\omega_{ij_lk} \geq 0$, $k \in \mathscr{V}(i)$, such that

$$\Delta\rho_{ij_l} = \sum_{k\in\mathscr{V}(i)} \omega_{ij_lk}\,\big(\rho_i - \rho_k\big), \qquad \text{with } \sum_{k\in\mathscr{V}(i)} \omega_{ij_lk} \leq \frac{7\tau}{12}\,C_{\mathscr{T}_h}.$$

Also, there exists for $l = 1,2$, some coefficients $\widetilde{\omega}_{ij_lk} \geq 0$, $k \in \mathscr{V}(i)$, such that

$$\widetilde{\Delta}\rho_{ij_l} = \sum_{k\in\mathscr{V}(i)} \widetilde{\omega}_{ij_lk}\,\big(\rho_k - \rho_i\big), \qquad \text{with } \sum_{k\in\mathscr{V}(i)} \widetilde{\omega}_{ij_lk} \leq 2.$$

Next, for $0 < \delta_{ij_l} < 1$, $l = 1,2$, we consider the following quantities:

$$\begin{aligned}\mathrm{E}_{ij_l} &= \frac{|\Gamma_{ij_l}^{(T)}|}{|\mathscr{C}_i|}\,\frac{\partial G_{ij_l}}{\partial\rho_1}\big(\rho_i + \delta_{ij_l}\Delta\rho_{ij_l},\ \rho_i + \delta_{ij_l}\widetilde{\Delta}\rho_{ij_l}\big), && l = 1,2,\\ \mathrm{F}_{ij_l} &= -\frac{|\Gamma_{ij_l}^{(T)}|}{|\mathscr{C}_i|}\,\frac{\partial G_{ij_l}}{\partial\rho_2}\big(\rho_i + \delta_{ij_l}\Delta\rho_{ij_l},\ \rho_i + \delta_{ij_l}\widetilde{\Delta}\rho_{ij_l}\big), && l = 1,2.\end{aligned}$$

Of course, by monotonicity of the numerical flux, we have $\mathrm{E}_{ij_l} \geq 0$ and $\mathrm{F}_{ij_l} \geq 0$. Hence, using the mean value theorem, the numerical scheme (11) is rewritten as follows:

$$\left(\mathbf{A}\rho^{n+1}\right)_i = \frac{4}{3}\rho_i^n - \frac{4}{3}\Delta t \sum_{T,M_i\in T}\sum_{k\in\mathscr{V}(i)}\sum_{l=1}^{2}\left(\omega_{ij_lk}\mathrm{E}^n_{ij_l}\left(\rho_i^n-\rho_k^n\right)-\widetilde{\omega}_{ij_lk}\mathrm{F}^n_{ij_l}\left(\rho_k^n-\rho_i^n\right)\right)$$
$$-\frac{1}{3}\rho_i^{n-1}+\frac{2}{3}\Delta t \sum_{T,M_i\in T}\sum_{k\in\mathscr{V}(i)}\sum_{l=1}^{2}\left(\omega_{ij_lk}\mathrm{E}^{n-1}_{ij_l}(\rho_i^{n-1}-\rho_k^{n-1})-\widetilde{\omega}_{ij_lk}\mathrm{F}^{n-1}_{ij_l}(\rho_k^{n-1}-\rho_i^{n-1})\right). \tag{12}$$

Finally, we obtain the following equations for each $i\in[1,I]$ and for all $n\geq 1$:

$$\left(\mathbf{A}\rho^{n+1}\right)_i = a_{ii}\,\rho_i^n + b_{ii}\,\rho_i^{n-1} + \sum_{k\in\mathscr{V}(i)}\left(a_{ik}\,\rho_k^n + b_{ik}\,\rho_k^{n-1}\right), \tag{13}$$

where a_{ii}, b_{ii}, a_{ik} and b_{ik} are easily determined from (12). Clearly, we have

$$a_{ii} + b_{ii} + \sum_{k\in\mathscr{V}(i)}\left(a_{ik} + b_{ik}\right) = 1. \tag{14}$$

Moreover, by choosing the time step Δt such that for all $i\in[1,I]$,

$$\Delta t \leq \left(\sum_{T,M_i\in T}\sum_{k\in\mathscr{V}(i)}\sum_{l=1}^{2}\left(\frac{4}{3}(\omega_{ij_lk}\mathrm{E}^n_{ij_l}+\widetilde{\omega}_{ij_lk}\mathrm{F}^n_{ij_l})-\frac{2}{3}(\omega_{ij_lk}\mathrm{E}^{n-1}_{ij_l}+\widetilde{\omega}_{ij_lk}\mathrm{F}^{n-1}_{ij_l})\right)\right)^{-1}, \tag{15}$$

we have

$$0 \leq a_{ii}+b_{ii} \leq 1 \qquad \text{and} \qquad 0 \leq a_{ik}+b_{ik} \leq 1. \tag{16}$$

Hence, (13), (14) and (16) allow us to conclude that for each $i\in[1,I]$, $\left(\mathbf{A}\rho^{n+1}\right)_i$ is written as convex combination of ρ_i^n, ρ_i^{n-1}, ρ_k^n and ρ_k^{n-1}, $k\in\mathscr{V}(i)$. □

Finally, as a consequence of Proposition 1, and recalling that an M-matrix is invertible with positive inverse, we obtain:

Theorem 1. *Let the velocity field* $\mathbf{u}$ *divergence free and the initial density* ρ_0 *such that* $\rho_0(x)\geq 0$. *Then, under the CFL condition (10) and the hypotheses (H1) and (H2) on the mesh, the linear system (9) is invertible, and*

$$\rho^{n+1}\geq 0, \qquad \forall\, n\geq 1. \tag{17}$$

Numerical results. Here we consider structured meshes on $\Omega =]-1,1[^2$, a stationary rotating velocity field $\mathbf{u}=(x_2,-x_1)$ and a small diffusion coefficient $\lambda = 10^{-6}$. Setting $r=\sqrt{(x_1+0.5)^2+x_2^2}$, the discontinuous initial condition is $\rho_0 = 1000$ if $r\leq 0.25$ and $\rho_0 = 1$ if $r>0.25$. The computations are performed for different values of $h\geq 0.004$, until $T=0.3$. In Fig. 2 we show the evolution of the density contours (left) and the solution profiles for some horizontal sections (right). We can remark that the maximum principle is well verified using the IMEX-BDF2 scheme, unlike other classical order two schemes, such as Crank-Nicolson Adams-Bashforth or Crank-Nicolson Runge-Kutta. Some other numerical results can be found in [7].

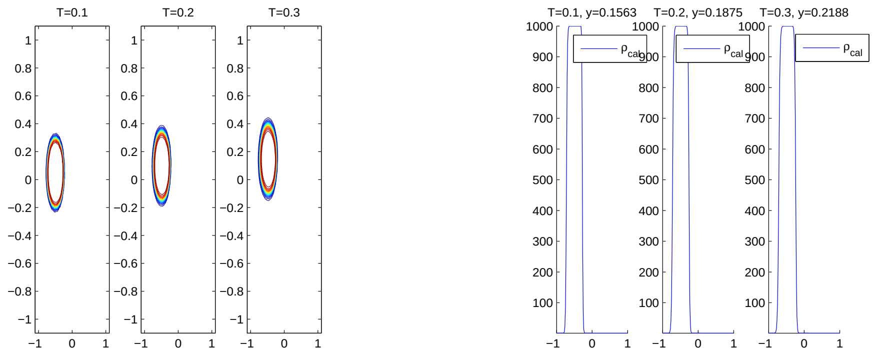

Fig. 2 The evolution of the density contours (left) and the solution profiles (right) for $\lambda = 10^{-6}$.

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